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Book reviews

Walter Van Assche and Carl de Boor, Book Review Editors

Books

Frank Deutsch, Best Approximation in Inner Product Spaces, in: CMS Books in Mathematics, Springer, New York, 2001 (xv +338pp, US$74.95, ISBN 0-387-95156-3).

Most of the books available today on approximation theory either do not cover best approximation in Hilbert spaces or hardly devote more than two chapters to the topic. This book comes to remedy the lack of a systematic treatment of this interesting subject. It does this in a pretty convincing way, providing comprehensive information on the problems of existence, uniqueness, characterization, continuity, computation, and error estimation of the best approximation. This is a self-contained textbook, written by one of the most distinguished experts in this field. The exposition is an equilibrated amalgam of rigorously presented theory, containing also the necessary prerequisites from functional analysis, and numerous pictures which help the reader to “see” the proofs. The book consists of 12 chapters and two appendices. Each chapter contains a list of exercises with gradually increasing difficulty and ends with very interesting historical notes complete with the necessary references.

The inner product spaces are introduced in the first chapter. The problem of best approximation is stated and initially discussed in the second chapter where five basic problems are also formulated. The questions of existence and uniqueness, and of characterization of the best approximation from convex sets and from subspaces are investigated in Chapters 3 and 4. The metric projection is under discussion in Chapter 5. Chapter 6 deals with best approximation from hyperplanes and half-spaces. Chapter 7 is devoted to the approximation error. It contains Kuhn’s elegant proof of the Weierstrass approximation theorem. Chapter 8, entitled “Generalized Solutions of Linear Equations”, treats mainly operator equations. Some basic results from functional analysis, such as the Baire, the Open Mapping, the Closed Range theorems are proved. No picture appears in this chapter though my experience shows that these proofs can also be visualized in order to facilitate the understanding for those who face these important theorems for the first time. Dykstra’s cyclic projections algorithm is discussed in Chapter 9, and Chapter 10 is devoted to

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constrained interpolation, in particular shape-preserving one. Chapter 11 is about simultaneous interpolation and approximation. The last chapter is my favorite and, unfortunately, the shortest one. It discusses one of the most challenging problems in approximation theory, namely, whether every Chebyshev set in a Hilbert space is convex. Since the author expects that the answer is negative, I would have preferred to see the example of a nonconvex Chebyshev set in an incomplete inner product space. The curious students probably would want this, too. Fortunately, the historical notes provide a comprehensive list of references.

Everyone who is interested in approximation theory should have a look at this excellently written book, especially those who are going to teach the subject next semester.

Dimitar K. Dimitrov
E-mail address: dimitrov@ibilce.unesp.br

Eli Levin, Doron S. Lubinsky, Orthogonal Polynomials for Exponential Weights, CMS Books in Mathematics, Springer, New York, 2001 (xi + 476pp, US$79.95, ISBN 0-387-98941-2).

The study of orthogonal polynomials has been an important part of classical analysis in the 20th century. Orthogonal polynomials have many different aspects, including explicit formulas, algebraic properties, asymptotics, and applications. The book under review is devoted to orthogonal polynomials with respect to a weight on a (finite or infinite) real interval, and in particular to the problem of finding asymptotic bounds and limits as the degree tends to infinity. It deals with general classes of weights characterized by growth and smoothness conditions, and as such it provides a substantial extension of the classical Szegő theory for weights w on $[-1, 1]$ that satisfy the Szegő condition $\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\, \mathrm{d}x > -\infty$. For such weights, Szegő developed very precise asymptotics of the orthonormal polynomials p_n as $n \to \infty$. This work is a milestone in the theory of orthogonal polynomials.

Typical examples of weights that do not satisfy the Szegő condition are the Pollaczek-type weights $w(x) = \exp(-2Q(x))$ with $Q(x) = (1 - x^2)^{-1/2}$ for $x \in (-1, 1)$. The Szegő theory also does not apply to weights on the real line such as the Freud weights, where $Q(x) = |x|^{\alpha}$ for $x \in \mathbf{R}$, or the Erdős weights where Q has faster than polynomial growth. These are canonical examples of what are called exponential weights in the book under review, and these are the objects of study. A common feature of exponential weights is that the orthogonal polynomial p_n 'lives' on an interval $[-a_n, a_n]$ that is strictly smaller than the interval of orthogonality (which is $(-1, 1)$ or $\mathbf{R}$). The numbers a_n are known as the Mhaskar–Rakhmanov–Saff numbers. They have an interpretation in potential theory since they are the endpoints of the support of an equilibrium measure in an external field.

Levin and Lubinsky studied the three cases mentioned above separately in the early 1990s. In three lengthy papers, they obtained uniform bounds on the orthogonal polynomials and mean and pointwise asymptotics on the real line (among many other results). A major achievement is the bound

$$\sup |p_n(x)|^2 w(x) \left|x^2 - a_n^2\right|^{1/2} \leqslant C,$$

where C is independent of n. The supremum is taken over all x in the interval of orthogonality.

The present book is the outcome of an ambitious project to give a unified treatment of these three cases, and at the same time to weaken the assumptions on the weights considerably. Most importantly, the authors remove the restriction that Q is an even function. Also the conditions on growth and smoothness are refined. The authors establish restricted range inequalities, Markov–Bernstein inequalities, estimates on Christoffel functions and equilibrium measures, bounds on orthogonal polynomials and their zeros, asymptotics for orthogonal polynomials, leading coefficients, recurrence coefficients, and mean and pointwise asymptotics on the interval of orthogonality.

This is an impressive book by two of the leading researchers in the field. By their nature, many proofs involve lengthy calculations and careful estimations. The authors have taken great care in explaining the basic steps and presenting the results in a clear and well-motivated way. It is very helpful that every chapter starts with a precise statement and discussion of the main results in that chapter. This is not an easy book, and it is not aimed at a beginner in the field. However, for any active researcher in the fascinating theory of asymptotics for orthogonal polynomials, it will be an indispensable source of inspiration.

Arno Kuijlaars
E-mail address: arno@wis.kuleuven.ac.be

N. Dyn, D. Leviatan, D. Levin, A. Pinkus (Eds.), Multivariate Approximation and Applications, Cambridge University Press, Cambridge, 2001 (x + 286pp, £55.-, ISBN-10:0521800234).

Multivariate approximation theory is an increasing active research area. It deals with a multitude of problems in areas such as multivariate interpolation, linear and nonlinear approximation schemes in high-dimensional spaces and manifolds, multiresolution analysis and wavelet analysis, multivariate splines, and bases for function spaces. Multivariate approximation theory has many applications to areas such as numerical analysis, signal processing and image compression, computer aided geometric design and computer graphics, numerical solutions to partial differential equations, and statistics. This book is an advanced introduction to multivariate approximation and related topics. It consists of nine articles written by leading experts surveying many of the new ideas and their applications.

The first part of this book, consisting of the first three chapters, is devoted to a study of multivariate scattered data interpolation. In this aspect radial basis functions have been proved to be an efficient tool. In Chapter 1, R. Schaback and H. Wendland introduce characterizations of (conditional) positive definiteness and give a new construction technique for radial basis functions. In Chapter 2, M. D. Buhmann provides a survey of recent developments on convergence rates of interpolation with radial basis functions. In Chapter 3,

H. N. Mhaskar, F. J. Narcowich, and J. D. Ward review various aspects of fitting surfaces to scattered data, addressing problems involving interpolation and order of approximation, and quadratures.

The second part of this book, consisting of Chapters 4 and 5, is concerned with shift-invariant spaces. Shift-invariant spaces play an increasingly important role in various areas of mathematical analysis and its applications. In Chapter 4, K. Jetter and G. Plonka give a self-contained introduction to notions and results connected with the L_2-approximation order of finitely generated shift-invariant spaces. In Chapter 5, A. Ron discusses generating sets for shift-invariant spaces and related topics such as linear independence and stability.

The third part of this book, consisting of Chapters 6 and 7, deals with two different aspects of wavelet analysis. In Chapter 6, T. Lyche, K. Mørken and E. Quak investigate mutually orthogonal spline wavelet spaces on nonuniform partitions of a bounded interval. In Chapter 7, A. Cohen provides a survey of nonlinear wavelet approximation from the perspective of its applications to data compression, statistical estimation and adaptive schemes for partial differential equations.

The fourth part of this book, consisting of Chapters 8 and 9, addresses applications of multivariate approximation theory to computer graphics. In Chapter 8, P. Schröder extends classical subdivision schemes to two new iterative algorithms: multiresolution surface editing and semi-regular remeshing. In Chapter 9, J. Hoschek describes mathematical methods in reverse engineering and surveys related algorithms including triangulation, segmentation, and B-spline approximation.

In summary, this book contains a selection of interesting survey papers on multivariate approximation and applications. The bibliography included in this book is comprehensive. It will serve as an excellent reference book for researchers and graduate students in mathematics, statistics, and computer sciences.

Rong-Qing Jia
E-mail address: rjia@ualberta.ca

Vladimir V. Andrievskii, Hans-Peter Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer Monographs in Mathematics, Springer, New York, 2002 (xiii + 438pp, US$89.95, ISBN 0-387-98652-9).

This monograph presents a modern, up-to-date view of a field of analysis namely polynomial approximation in the complex plane. Development of the classical results such as: Jentzsch’s and Szegő’s theorems on limit points of zeros of partial sums of power series, Bernstein’s theorem on the characterisation of an analytic continuation of a function by means of the behaviour of the zeros of their best polynomial approximants, Walsh’s results about connection of the distribution of good interpolation points for polynomial approximation with the equilibrium measure of the set where the function has to be approximated—leads to more delicate problems about the behaviour of extremal discrete measures in the complex plane and their limits. The main direction which is investigated in the monograph is to find discrepancy estimates for signed measures supported on curves or arcs in the complex plane assuming that bounds for their logarithmic potential or energy functional are known as input data. Typically, the positive component of the signed measure has to take over the role of an approximation to the equilibrium measure of the given curve or arc. On the other hand, in practical applications, the role of the negative component is taken over by discrete zeros counting measures of polynomials. The main tools applied here are a nice combination of modern potential theory with the geometric theory of complex analysis, including conformal invariants, the module of a family of curves, and quasiconformal curves.

The book consists of eight chapters (each of them provided with valuable historical comments) and four appendices. The bibliography contains 184 references.

The first two chapters have an introductory character. There, auxiliary facts and classical theorems about the distribution of zeros of polynomial approximants are presented. All necessary concepts from potential theory, conformal and quasiconformal mapping necessary for further understanding as well as theorems of Jentzsch–Szegő type and theorems of Erdős–Turán are given here.

The next three chapters are the bulk of the monograph. There, results of the authors on discrepancy theorems are presented. Chapter 3 contains discrepancy theorems via two-sided bounds on potentials. Chapter 4 uses one-sided bounds on potentials, and the derivation of discrepancy theorems via energy integrals is in Chapter 5.

The last two chapters contain applications of Jentzsch–Szegő (and Erdős–Turán) type theorems as well as applications of discrepancy theorems. Here, among others, the following topics are considered: polynomials of best and near-best approximation, distribution of Fekete points and of alternation points in Chebyshev approximation, zeros of orthogonal polynomials, and polynomial approximation to a piecewise analytic function on touching domains.

In the appendices, some details from background material are the focus, in particular conformally invariant characteristics of curve families (module, extremal length), and basics in the theory of quasiconformal mappings and in the constructive theory of functions of a complex variable.

Overall, this is a valuable book for those wishing to learn about distributions of zeros of polynomial sequences and their application to approximation theory in the complex plane. Good book. I like it.

Alexander I. Aptekarev
E-mail address: aptekaa@keldysh.ru

Ole Christensen, Khadija L. Christensen, Approximation Theory. From Taylor Polynomials to Wavelets, Applied and Numerical Harmonic Analysis, Birkhäuser, Basel, 2004 (xi + 156pp, US$34.95, ISBN 0-8176-3600-5).

To quote from the Preface: “This book gives an elementary introduction to a classical area of mathematics—Approximation Theory—in a way that naturally leads to the modern field of wavelets. …The focus is on ideas rather than on technical details, and the book is not primarily meant as a textbook.”

Approximation Theory does not get beyond Taylor, Fourier, and Weierstrass, but the discussion of wavelets and frames is, of necessity, more up to date.

Chapter headings:
1. Approximation with Polynomials;
2. Infinite Series;
3. Fourier Analysis;
4. Wavelets and Applications;
5. Wavelets and their Mathematical Properties.

There are also two appendices, providing proofs of some of the basic results stated in the text.

C. de Boor
E-mail address: deboor@cs.wisc.edu

Ole Christensen, An Introduction to Frames and Riesz Bases, Applied and Numerical Harmonic Analysis, Birkhäuser, Basel, 2003 (xx + 440pp, US$69.95, ISBN-0-8176-4295-1).

General The notion of a *basis* plays a central role in linear algebra. This notion admits several different worthy generalizations to the infinite-dimensional setup. The mathematical study of such bases occupies an important chapter in Functional Analysis, Approximation Theory, and Harmonic Analysis.

In contrast, overcomplete representations, i.e., representations in terms of a fundamental set of vectors that do not necessarily form a basis, have traditionally received far less attention within Mathematical Analysis. This was particularly true in the emerging area of wavelet representations, with research in that area, during its early years of development, focusing almost exclusively on representations in terms of orthonormal and Riesz bases. This led to the dichotomy where the published research in the area was devoted mostly to exact representations, while the majority of the practical algorithms employed overcomplete, aka redundant, representations.

The last decade witnessed a significant change in the field of data representation, with the theory and applications of redundant representations taking center stage and becoming a central research topic in the areas of wavelet and Gabor representations. The specific topic of *frame representations* received particular attention and became a major theme for these efforts. The book in review successfully summarizes that progress. Some of its chapters are basic, and are suitable for use in a graduate course in Mathematics. Other chapters provide the specialist with a detailed up-to-date review of the state-of-the-art in the field. Other scientists, with more general interest in the area, might use the book as a general reference on the topic.

Definition of frames For the case of a separable Hilbert space H with norm $\|\cdot\|$, a *frame* X is a countable subset of H that satisfies, for some positive constants A, B,

$$A\|h\|^2 \leqslant \sum_{x\in X} |\langle h, x\rangle|^2 \leqslant B\|h\|^2, \quad \text{all } h \in H.$$

The frame X is *tight* if the above holds for $A = B = 1$. Thus, every Riesz basis for H is a frame, while every orthonormal basis for H is a tight frame. The premise of each and all studies on frames is that it is easier, sometime far easier, to construct a frame with customized properties as compared to the construction of a basis with such properties. The theories of framelets (i.e., wavelet frames) and Gabor frames provide, in turn, overwhelming evidence for the validity of the aforementioned premise.

Organization and content of the book The book consists of 17 chapters that can be grouped into six parts.

I. Preliminaries—Chapters 1 and 2: Concepts covered are frames in finite-dimensional spaces, the rudiments of Banach and Hilbert space theory, the Fourier transform, and the space $L_2(\mathbb{R})$. An Appendix contains miscellanea including a brief description of B-splines.

II. General discussion of bases and frames—Chapters 3, 4, 5 and 6: *Bases* are introduced gradually with the notion of Schauder bases followed by Riesz bases, unconditional bases and orthonormal bases. The treatment of frames includes a comparison with bases as well as a discussion of the canonical dual frame and general dual frames.

III. Frames of translates—Chapter 7: The chapter discusses frames for $L_2(\mathbb{R})$ or a subspace of it whose elements are translates of a finite set of functions. The discussion addresses both the principal shift-invariant case (where the lattice translates of a single function are involved) as well as the case of irregular translates.

IV. Gabor frames—Chapters 8, 9 and 10: The presentation on Gabor frames is detailed. It includes the characterization of Gabor frames of $L_2(\mathbb{R})$, several sufficient and necessary conditions, the Zak transform, the duality principles, as well as a few important identities.

V. Wavelet frames—Chapters 11, 12, 13 and 14: After covering the continuous (hence completely redundant) wavelet transform, the author describes the historical evolution of the subject. One chapter is devoted to the non-constructive theories that existed in the area before the birth of multiresolution analysis (MRA). A second chapter analyzes a frame construction methodology that follows closely the original MRA framework of Mallat and Meyer. The last chapter in this evolution covers the construction of framelets by extension principles: the Unitary Extension Principle and the Oblique Extension Principle. (The book does not, and actually could not, cover the very recent accomplishments in this area, i.e., the CAP and CAMP constructions).

VI. Miscellaneous topics—Chapters 15, 16 and 17: These include perturbation of frames, a treatment of the inverse frame operators, and discussion of frames in more general contexts: on locally compact groups and in Banach spaces.

Amos Ron
E-mail address: amos@cs.wisc.edu

J. Michael Steele, The Cauchy–Schwarz Master Class, Math. Assoc. America, Washington, DC, 2004 (320pp, US$36.-, ISBN D 521-54677X).

According to the book's subtitle, the author offers "An Introduction to the Art of Mathematical Inequalities". The book is written in the style of Pólya's classic "How to solve it".

Much attention is paid to presenting possibly the shortest and simplest proofs of various inequalities. In addition, the reader can also learn why certain seemingly natural approaches fail. The book could be a pleasant reading for everyone with a solid real analysis background at undergraduate level, even before reading Pólya–Szegő. In fact, even researchers working on topics close to those in this book can find much to add to their repertoire. The author offers standard discussions as well as less standard ones of well-known classical inequalities such as the Cauchy–Schwarz a.k.a. Bunyakovskii (Cauchy was first, and stated it for sums; Bunyakovskii was next, and stated it for integrals; Schwarz was third, and stated it for arbitrary inner products), Minkowski, Hölder, Rogers, Chebyshev's Order, AM-GM, Power Mean, Jensen, Carleman, Abel, Newton–Maclaurin, van der Corput, Hardy, Hilbert, Muirhead, Young inequalities and some of their applications. However, the book is special for some other reasons. The book deals with many choice problems the solutions to which require more than a routine application of the theoretical background of the given section. An example for this is a discussion of the solution to an American Mathematical Monthly problem proposed by M. Mazur, where it is demonstrated quite transparently that an effective use of Jensen's inequality calls for one to find a function that is convex on the positive real axis and that is never larger than a given function f, see Fig. 6.3. I particularly like Pólya's proof of the celebrated Carleman inequality. Section 10 was also a highlight for me. Proving that π in Hilbert's Inequality is the best possible, while 4 is the best constant in its max version is exceptionally educational. I have learned much from Section 12 about symmetric sums. The proof of the Newton–Maclaurin inequalities presented here is really beautiful. Section 13 is about majorization and Schur convexity, which are viewed by the author as two of the most productive concepts in the theory of inequalities. Indeed, they unify the understanding of many familiar bounds. Since they are not as well known as they should be, they can become one's secret weapon. A nice treatment of Birkhoff's Theorem on doubly stochastic matrices shows connections to areas such as probability and operator theory, while the proof of the "marriage lemma" given by Halmos and Vaughan in 1950 shows relations to discrete mathematics. This "marriage lemma" is a cornerstone of the large and active field of matching theory which is beautifully surveyed by Lovász and Plummer (1986).

The historical component of the book is also remarkable. The aim of this paragraph is to illustrate where the presentation of a topic in this book starts and ends. The problem whether it is always possible to write a real polynomial of d variables that is nonnegative on its domain as a sum of squares of some other polynomials of d variables, turns out to be wonderfully rich. When $d = 1$ the answer is yes, and this can be shown rather simply, but it was Hilbert who first proved that this is not always possible when $d \geqslant 2$, as it was conjectured earlier by Minkowski. Hilbert's proof was long, subtle, and indirect. The first explicit example of a nonnegative polynomial that cannot be written as the sum of the squares of real polynomials was given in 1967, almost 80 years after Hilbert proved the existence of such polynomials. The explicit example this book presents was discovered by T. S. Motzkin. The 17th problem of Hilbert's great list is a direct descendant of Minkowski's conjecture. In this problem Hilbert asked whether every nonnegative real polynomial in d variables has a representation as a sum of squares of ratios of polynomials. This modification of Minkowski's problem makes a difference, and Hilbert's question was answered affirmatively in 1927 by Emil Artin. Artin's solution to Hilbert's 17th problem is now widely considered to be one of the crown jewels of modern algebra, but it is clearly beyond the scope of this book.

Each section contains several interesting and challenging exercises. Even when some of these do not prove to be some of the most important problems of the day, the reader has a chance to learn their short and elegant solutions in the closing section. In this respect the author must have followed Pólya–Szegő. Participants in the 2002 Canadian Mathematics Olympiad may find not only a familiar problem, Exercise 2.4, but an arsenal to prove inequalities which would be far too difficult even in the International Mathematics Olympiad, and not only because of the time constraint. A large mathematics department with a functional graduate program could easily consider to offer a master course based on this book.

Tamás Erdélyi
E-mail address: terdelyi@math.tamu.edu

Collections and Proceedings

Special Functions, Charles Dunkl, Mourad Ismail, and Roderick Wong, eds., World Scientific, Singapore, 2000, xi + 438pp.

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Approximation Theory X: Abstract and Classical Analysis, C. K. Chui, L. L. Schumaker, and J. Stöckler, eds., Vanderbilt University Press, Nashville, 2002, xiv + 402pp.

Approximation Theory X: Splines, Wavelets, and Applications, C. K. Chui, L. L. Schumaker, and J. Stöckler, eds., Vanderbilt University Press, Nashville, 2002, xv + 483pp.

These two volumes are the proceedings of the 10th International Symposium on Approximation Theory, held 26–29 March 2001 at the Sheraton Conference Center in Westport Plaza, St. Louis.

Curve and Surface Fitting: Saint-Malo 2002, Albert Cohen, Jean-Louis Merrien, and Larry L. Schumaker, eds., Nashboro Press, Brentwood TN, 2003, xi + 402pp.

Curve and Surface Design: Saint-Malo 2002, Tom Lyche, Marie-Laurence Mazure, and Larry L. Schumaker, eds., Nashboro Press, Brentwood TN, 2003, xi + 416pp.

These two volumes are the proceedings of the Fifth International Conference on Curves and Surfaces, held 27 June–3 July 2002 at the Palais du Grand Large in Saint-Malo, France.

Approximation Theory: A Volume Dedicated to Blagodest Sendov, B. D. Bojanov, ed., DARBA, Sofia, 2002, xx + 410pp.

Constructive Theory of Functions, Varna 2002, B. D. Bojanov, ed., DARBA, Sofia, 2002, xxii + 452pp.

This volume contains the proceedings of the Varna conference held 19–23 June 2002 and dedicated to the 70th birthday of Blagodest Sendov.